

MONIC REPRESENTATIONS AND GORENSTEIN-PROJECTIVE MODULES

XIU-HUA LUO, PU ZHANG*

Dedicated to the memory of Hua Feng

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, P. R. China

ABSTRACT. Let Λ be the path algebra of a finite quiver Q over a finite-dimensional algebra A . Then Λ -modules are identified with representations of Q over A . This yields the notion of monic representations of Q over A . If Q is acyclic, then the Gorenstein-projective Λ -modules can be explicitly determined via the monic representations. As an application, A is self-injective if and only if the Gorenstein-projective Λ -modules are exactly the monic representations of Q over A .

Key words and phrases. representations of a quiver over an algebra, monic representations, Gorenstein-projective modules

1. Introduction

Let A be an Artin algebra, and $A\text{-mod}$ the category of finitely generated left A -modules. A complete A -projective resolution is an exact sequence of finitely generated projective A -modules

$$P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

such that $\text{Hom}_A(P^\bullet, A)$ remains to be exact. A module $M \in A\text{-mod}$ is *Gorenstein-projective*, if there exists a complete A -projective resolution P^\bullet such that $M \cong \text{Ker } d^0$. Let $\mathcal{P}(A)$ be the full subcategory of $A\text{-mod}$ of projective modules, and $\mathcal{GP}(A)$ the full subcategory of $A\text{-mod}$ of Gorenstein-projective modules. Then $\mathcal{P}(A) \subseteq \mathcal{GP}(A) \subseteq {}^\perp A = \{X \in A\text{-mod} \mid \text{Ext}_A^i(X, A) = 0, \forall i \geq 1\}$. It is clear that $\mathcal{GP}(A) = A\text{-mod}$ if and only if A is self-injective. If A is of finite global dimension then $\mathcal{GP}(A) = \mathcal{P}(A)$; and if A is a *Gorenstein algebra* (i.e., $\text{inj.dim } {}_A A < \infty$ and $\text{inj.dim } A_A < \infty$), then $\mathcal{GP}(A) = {}^\perp A$ ([EJ], Corollary 11.5.3). This class of modules enjoys more stable properties than the usual projective modules ([AB], where it was called a module of G -dimension zero); it become an important ingredient in the relative homological algebra ([EJ]) and in the representation theory of algebras (see e.g. [AR], [B], [GZ], [IKM]); and plays a central role in the Tate cohomology of algebras (see e.g. [AM] and [Buch]). By [Buch] and [Hap], the singularity category of Gorenstein algebra A is triangle equivalent to the stable category of Gorenstein-projective A -modules.

*The corresponding author.

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X.H.Luo@sjtu.edu.cn pzhang@sjtu.edu.cn.

On the other hand, the submodule category have been extensively studied in [RS1] (see also [RS2], [S]). By [KLM] it is also related to the singularity category (see also [C]). It turns out that the category of the Gorenstein-projective modules is closely related to the submodule category, or in general, to the monomorphism category (see [Z]). The present paper is to explore such an relation in a more general setting up.

Let Λ be the path algebra of a finite quiver Q over A , where A is a finite-dimensional algebra over a field k . As in the case of $A = k$, Λ -modules can be interpreted as representations of Q over A . This interpretation permits us to introduce the so-called monic representations of Q over A . If $Q = \bullet_2 \rightarrow \bullet_1$ then they are exactly the objects in the submodule category of A (see [RS1]); and if $Q = \bullet_n \rightarrow \cdots \rightarrow \bullet_1$ then the monic representations of Q over A are exactly the objects in the monomorphism category of A ([Z]). The main result Theorem 4.1 of this paper explicitly determine all the Gorenstein-projective Λ -modules when Q is acyclic (i.e., Q has no oriented cycles), via the monic representations of Q over A . We emphasize that here Λ is not necessarily Gorenstein. The proof of Theorem 4.1 use induction on $|Q_0|$, and a description of the Gorenstein-projective modules over the triangular extension of two algebras via bimodules which is projective in both sides (Theorem 3.1). As an application, we get a characterization of self-injectivity by claiming that A is self-injective if and only if the Gorenstein-projective Λ -modules are exactly the monic representations of Q over A (Theorem 5.1). As another consequence, if Q has an arrow, then the projective Λ -modules coincide with the monic representations of Q over A if and only Λ is hereditary (Theorem 5.4).

2. Monic representations of a quiver over an algebra

Throughout this section k is a field, Q a finite quiver, and A a finite-dimensional k -algebra. We consider the path algebra AQ of Q over A , describe its module category, and introduce the so-called the monic representations of Q over A .

2.1. For the notion of a finite quiver $Q = (Q_0, Q_1, s, e)$ we refer to [ARS] and [R]. We write the conjunction of paths of Q from right to left. Let \mathcal{P} be the set of paths of Q . Vertex i is a path of length 0 and denote it by e_i . We define $s(e_i) = i = e(e_i)$. If $p = \alpha_l \cdots \alpha_1 \in \mathcal{P}$ with $\alpha_i \in Q_1$, $l \geq 1$, and $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq l-1$, then we call l the length of p and denote it by $l(p)$, and define the starting vertex $s(p) = s(\alpha_1)$, and the ending vertex $e(p) = e(\alpha_l)$. Let kQ be the path algebra of Q over k . It is well-known that the category $kQ\text{-mod}$ of finite-dimensional kQ -modules is equivalent to the category $\text{Rep}(Q, k)$ of finite-dimensional representations of Q over k .

2.2. Let $\Lambda = AQ$ be the free left A -module with basis \mathcal{P} . An element of AQ is written as a finite sum $\sum_{p \in \mathcal{P}} a_p p$, where $a_p \in A$ and $a_p = 0$ for all but finitely many p . Then Λ has a k -algebra structure, with multiplication bi-linearly given by $(a_p p)(b_q q) = (a_p b_q)(pq)$, where $a_p b_q$ is the product in A , and pq is the product in kQ . We have isomorphisms $\Lambda \cong A \otimes_k kQ \cong kQ \otimes_k A$ of k -algebras, and we call $\Lambda = AQ$ the path algebra of Q over A .

For example, if $Q = \bullet_n \longrightarrow \cdots \longrightarrow \bullet_1$ then Λ is $T_n(A) = \begin{pmatrix} A & A & \cdots & A & A \\ 0 & A & \cdots & A & A \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A & A \\ 0 & 0 & \cdots & 0 & A \end{pmatrix}$, the upper triangular matrix algebra of A . In general, if Q is acyclic, then Λ is also a kind of upper triangular matrix algebra over A . More precisely, we label Q_0 as $1, \dots, n$, such that if there is an arrow $\alpha : j \longrightarrow i$ in Q_1 then $j > i$. Then kQ is isomorphic to the following matrix algebra over k :

$$\begin{pmatrix} k & k^{m_{21}} & k^{m_{31}} & \cdots & k^{m_{n1}} \\ 0 & k & k^{m_{32}} & \cdots & k^{m_{n2}} \\ 0 & 0 & k & \cdots & k^{m_{n3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix}_{n \times n}, \quad (2.1)$$

where m_{ji} is the number of paths from j to i , and $k^{m_{ji}}$ is the direct sum of m_{ji} copies of k . It follows that Λ is isomorphic to the following matrix algebra over A :

$$\begin{pmatrix} A & A^{m_{21}} & A^{m_{31}} & \cdots & A^{m_{n1}} \\ 0 & A & A^{m_{32}} & \cdots & A^{m_{n2}} \\ 0 & 0 & A & \cdots & A^{m_{n3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix}_{n \times n}. \quad (2.2)$$

2.3. By definition, a representation X of Q over A is a datum $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$, where X_i is an A -module for each $i \in Q_0$, and $X_\alpha : X_{s(\alpha)} \longrightarrow X_{e(\alpha)}$ is an A -map for each $\alpha \in Q_1$. It is a *finite-dimensional representation* if each X_i is finite-dimensional. We will call X_i the *i -th branch* of X . A morphism f from representation X to representation Y is a datum $(f_i, i \in Q_0)$, where $f_i : X_i \longrightarrow Y_i$ is an A -map for each $i \in Q_0$, such that for each arrow $\alpha : j \longrightarrow i$ the following diagram

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & Y_j \\ \downarrow X_\alpha & & \downarrow Y_\alpha \\ X_i & \xrightarrow{f_i} & Y_i \end{array} \quad (2.3)$$

commutes. If $p = \alpha_l \cdots \alpha_1 \in \mathcal{P}$ with $\alpha_i \in Q_1$, $l \geq 1$, and $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq l-1$, then we put X_p to be the A -map $X_{\alpha_l} \cdots X_{\alpha_1}$. Denote by $\text{Rep}(Q, A)$ the category of finite-dimensional representations of Q over A . A morphism $f = (f_i, i \in Q_0)$ in $\text{Rep}(Q, A)$ is a monomorphism (resp., an epimorphism, an isomorphism) if and only if for each $i \in Q_0$, f_i is injective (resp., surjective, an isomorphism).

Lemma 2.1. *Let Λ be the path algebra of Q over A . Then we have an equivalence $\Lambda\text{-mod} \cong \text{Rep}(Q, A)$ of categories, here $\Lambda\text{-mod}$ is the category of finite-dimensional Λ -modules.*

We omit the details of the proof of Lemma 2.1, which is similar to the case of $A = k$ (see Theorem 1.5 of [ARS], p.57; or [R], p.44). In the following we will identify a Λ -module with a representation of Q over A , which is always assumed to be finite-dimensional. Under this identification, a Λ -module X is a representation $(X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over A , where $X_i = (1e_i)X$, 1 is the identity of A , and the A -action on X_i is given by $a(1e_i)x = (ae_i)x = (1e_i)(ae_i)x$, $\forall x \in X, \forall a \in A$; and $X_\alpha : X_{s(\alpha)} \longrightarrow X_{e(\alpha)}$ is the A -map given by the left action by $1\alpha \in \Lambda$. On the other hand,

a representation $(X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over A is a Λ -module $X = \bigoplus_{i \in Q_0} X_i$, with the Λ -action on X given by

$$(ap)(x_i) = \begin{cases} 0, & \text{if } s(p) \neq i; \\ ax_i, & \text{if } s(p) = i, l(p) = 0; \\ aX_p(x_i) \in X_{e(p)}, & \text{if } s(p) = i, l(p) \geq 1, \end{cases} \quad \forall a \in A, \forall p \in \mathcal{P}, \forall x_i \in X_i.$$

An indecomposable projective Λ -module is of the form $L \otimes_k P(i)$, where $P(i)$ is the indecomposable kQ -module at vertex i , and L is an indecomposable projective A -module. In particular, each branch of a projective Λ -module is a projective A -module.

Let $f : X \rightarrow Y$ be a morphism in $\text{Rep}(Q, A)$. Then $\text{Ker } f$ and $\text{Coker } f$ can be explicitly written out. For example, $\text{Coker } f = (\text{Coker } f_i, \widetilde{Y}_\alpha, i \in Q_0, \alpha \in Q_1)$, where for each arrow $\alpha : j \rightarrow i$, $\widetilde{Y}_\alpha : \text{Coker } f_j \rightarrow \text{Coker } f_i$ is the A -map induced by Y_α (see (2.3)). A sequence of morphisms $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ in $\text{Rep}(Q, A)$ is exact if and only if $0 \rightarrow X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \rightarrow 0$ is exact in $A\text{-mod}$ for each $i \in Q_0$.

In the following, if Q_0 is labeled as $1, \dots, n$, then we also write a representation X of Q over A as $\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(X_\alpha, \alpha \in Q_1)}$.

2.4. The following is a central notion in this paper.

Definition 2.2. A representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over A is a *monic representation*, or a *monic Λ -module*, if for each $i \in Q_0$ the following A -map

$$(X_\alpha)_{\alpha \in Q_1, e(\alpha)=i} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \rightarrow X_i$$

is injective, or equivalently, the following two conditions are satisfied

(m1) For each $\alpha \in Q_1$, $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$ is an injective map; and

(m2) For each $i \in Q_0$, there holds $\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha = \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha$.

Denote by $\text{Mon}(Q, A)$ the full subcategory of $\text{Rep}(Q, A)$ consisting of the monic representations of Q over A . In particular, if $A = k$, then we have $\text{Mon}(Q, k) \subseteq \text{Rep}(Q, k)$.

For example, if $Q = \bullet_n \rightarrow \dots \rightarrow \bullet_1$ then a representation X of Q over A is simply written as $X = (X_i, \phi_i)$, where each $\phi_i : X_{i+1} \rightarrow X_i$ is an A -map, $1 \leq i \leq n-1$. In this case X is a monic representation, or a monic $T_n(A)$ -module, exactly means that each ϕ_i is injective, $1 \leq i \leq n-1$. This kind of monic $T_n(A)$ -modules have arisen from different questions and in different terminologies, for examples in [RS1], [RS2], [S], [LZ], [KLM], [C], [Z], [IKM].

2.5. There is another similar but different notion. Let $A = kQ/I$ be a finite-dimensional k -algebra, where I is an admissible ideal of kQ . An I -bounded representations of Q over k is a datum

$X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$, where X_i is a k -space for each $i \in Q_0$, and $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$ is a k -linear map for each $\alpha \in Q_1$, such that $\sum_{p \in \mathcal{P}} c_p X_p = 0$ for each element $\sum_{p \in \mathcal{P}} c_p p \in I$, where $l(p) \geq 2$ and $c_p \in k$. An I -bounded representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over k is a *monic representation*, if for each $i \in Q_0$ the following k -linear map

$$(X_\alpha)_{\alpha \in Q_1, e(\alpha)=i} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \rightarrow X_i$$

is injective. Let $\text{Rep}(Q, I, k)$ be the category of finite-dimensional I -bounded representations of Q over k . It is well-known that there is an equivalence $A\text{-mod} \cong \text{Rep}(Q, I, k)$ of categories (see Proposition 1.7 in [ARS], p.60; or [R], p.45). Let $\text{Mon}(Q, I, k)$ denote the full subcategory of $\text{Rep}(Q, I, k)$ of I -bounded monic representations Q over k . Thus $\text{Mon}(Q, 0, k) = \text{Mon}(Q, k)$.

Proposition 2.3. *Let $A = kQ/I$ be a finite-dimensional k -algebra, where I is an admissible ideal of kQ . Then $\mathcal{P}(A) \subseteq \text{Mon}(Q, I, k)$ if and only if A is hereditary.*

Proof. If A is hereditary then $I = 0$. It is clear $\mathcal{P}(kQ) \subseteq \text{Mon}(Q, 0, k)$.

Conversely, if $I \neq 0$, then take an element $\sum_{p \in \mathcal{P}} c_p p \in I$ with $l(p) \geq 2$ and $c_p \in k$. Assume that all the paths p with $c_p \neq 0$ have the same starting vertex j and the same ending vertex i . Consider the projective A -module $P(j) = Ae_j$. As an I -bounded representation of Q over k we have $P(j) = (e_t k Q e_j, t \in Q_0, f_\alpha, \alpha \in Q_1)$. Let $\alpha_1, \dots, \alpha_m$ be all the arrows of Q ending at i . We claim that

$$(f_{\alpha_v})_{1 \leq v \leq m} : \bigoplus_{1 \leq v \leq m} e_{s(\alpha_v)} k Q e_j \rightarrow e_i k Q e_j$$

is not injective, where f_{α_v} is the k -linear map given by the left multiplication by α_v . Since each path from j to i must go through some α_v , and $\sum_{p \in \mathcal{P}} c_p f_p = 0$, it follows that $\sum_{1 \leq v \leq m} \dim_k(e_{s(\alpha_v)} k Q e_j) > \dim_k(e_i k Q e_j)$. This justifies the claim, i.e., $P(j) \notin \text{Mon}(Q, I, k)$. \blacksquare

Now, let $\Lambda = A \otimes_k kQ$ be the path algebra of Q over A . Assume that Λ is of the form $\Lambda = kQ'/I'$, where Q' is a finite quiver and I' is an admissible ideal of kQ' . We emphasize that in general $\text{Mon}(Q, A) \neq \text{Mon}(Q', I', k)$. In fact, we will see in Theorem 4.1 that $\mathcal{P}(\Lambda) \subseteq \text{Mon}(Q, A)$ is always true; but in general $\mathcal{P}(\Lambda) \subseteq \text{Mon}(Q', I', k)$ is not true, as Proposition 2.3 shows. This is the reason why we do not use the notation $\text{Mon}(\Lambda)$.

3. Algebras given by bimodules

3.1. Let A and B be rings, and M an A - B -bimodule. Consider the upper triangular matrix ring $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where the addition and the multiplication are given by the ones of matrices. We assume that Λ is an Artin algebra ([ARS], p.72), and only consider finitely generated Λ -modules. A Λ -module can be identified with a tripe $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$, or simply $\begin{pmatrix} X \\ Y \end{pmatrix}$ if ϕ is clear, where $X \in A\text{-mod}$, $Y \in B\text{-mod}$, and $\phi : M \otimes_B Y \rightarrow X$ is an A -map. A Λ -map $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \rightarrow \begin{pmatrix} X' \\ Y' \end{pmatrix}_{\phi'}$ can be identified

with a pair $\begin{pmatrix} f \\ g \end{pmatrix}$, where $f \in \text{Hom}_A(X, X')$, $g \in \text{Hom}_B(Y, Y')$, such that the diagram

$$\begin{array}{ccc} M \otimes_B Y & \xrightarrow{\phi} & X \\ \text{id} \otimes g \downarrow & & \downarrow f \\ M \otimes_B Y' & \xrightarrow{\phi'} & X' \end{array}$$

commutes. A sequence of Λ -maps $0 \rightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\phi_1} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\phi_2} \xrightarrow{\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\phi_3} \rightarrow 0$ is exact if and only if $0 \rightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \rightarrow 0$ is an exact sequence of A -maps, and $0 \rightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \rightarrow 0$ is an exact sequence of B -maps. Indecomposable projective Λ -modules are exactly $\begin{pmatrix} P \\ 0 \end{pmatrix}$ and $\begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}_{\text{id}}$, where P runs over indecomposable projective A -modules, and Q runs over indecomposable projective B -modules.

3.2. The following result describes the Gorenstein-projective Λ -modules, if ${}_A M$ and M_B are projective modules. We emphasize that here Λ is not assumed to be Gorenstein (see Corollary 3.3 of [XZ] for the similar result under the assumption that Λ is Gorenstein; and the proof there in [XZ] can not be generalized to the non-Gorenstein case).

Theorem 3.1. *Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an Artin algebra, M an A - B -bimodule such that ${}_A M$ and M_B are projective modules. Then $\begin{pmatrix} X \\ Y \end{pmatrix}_{\phi} \in \mathcal{GP}(\Lambda)$ if and only if $\phi : M \otimes_B Y \rightarrow X$ is injective, $\text{Coker } \phi \in \mathcal{GP}(A)$, and $Y \in \mathcal{GP}(B)$. In this case, $X \in \mathcal{GP}(A)$ if and only if $M \otimes_B Y \in \mathcal{GP}(A)$.*

Proof. The last assertion is easy, since in this case $0 \rightarrow M \otimes_B Y \xrightarrow{\phi} X \rightarrow \text{Coker } \phi \rightarrow 0$ is exact, and $\mathcal{GP}(A)$ is closed under extensions and the kernels of epimorphisms (see e.g. [Hol]).

We first prove the “if” part. Assume that $\phi : M \otimes_B Y \rightarrow X$ is injective, $\text{Coker } \phi \in \mathcal{GP}(A)$, and $Y \in \mathcal{GP}(B)$. Then we have a complete B -projective resolution

$$Q^\bullet = \cdots \rightarrow Q^{-1} \rightarrow Q^0 \xrightarrow{d'^0} Q^1 \rightarrow \cdots \quad (3.1)$$

with $Y = \text{Ker } d'^0$, and a complete A -projective resolution

$$P^\bullet = \cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots \quad (3.2)$$

with $\text{Coker } \phi = \text{Ker } d^0$. Since M_B is projective, we get the following exact sequences of A -modules

$$\begin{aligned} 0 \rightarrow M \otimes_B Y \rightarrow M \otimes_B Q^0 \rightarrow M \otimes_B Q^1 \rightarrow \cdots \\ 0 \rightarrow \text{Coker } \phi \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots \end{aligned}$$

Since ${}_A M$ is projective, $M \otimes_B Q^i$ is a projective A -module for each $i \geq 0$. Note that projective A -modules are injective objects in $\mathcal{GP}(A)$, it follows from the exact sequence $0 \rightarrow M \otimes_B Y \rightarrow X \rightarrow \text{Coker } \phi \rightarrow 0$ and a version of Horseshoe Lemma that there is an exact sequence of A -modules

$$0 \rightarrow X \rightarrow P^0 \oplus (M \otimes_B Q^0) \xrightarrow{\partial^0} P^1 \oplus (M \otimes_B Q^1) \rightarrow \cdots \quad (3.3)$$

with $\partial^i = \begin{pmatrix} d^i & 0 \\ \sigma^i & \text{id} \otimes_B d'^i \end{pmatrix}$, $\sigma^i : P^i \rightarrow M \otimes_B Q^i$, $\forall i \in \mathbb{Z}$, such that the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_B Y & \longrightarrow & M \otimes_B Q^0 & \xrightarrow{\text{id} \otimes_B d'^0} & M \otimes_B Q^1 \longrightarrow \dots \\
& & \downarrow \phi & & \downarrow \binom{0}{\text{id}} & & \downarrow \binom{0}{\text{id}} \\
0 & \longrightarrow & X & \longrightarrow & P^0 \oplus (M \otimes_B Q^0) & \xrightarrow{\partial^0} & P^1 \oplus (M \otimes_B Q^1) \longrightarrow \dots
\end{array} \quad (3.4)$$

commutes. By the same argument we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\dots & \longrightarrow & M \otimes_B Q^{-2} & \xrightarrow{\text{id} \otimes_B d'^{-2}} & M \otimes_B Q^{-1} & \longrightarrow & M \otimes_B Y \longrightarrow 0 \\
& & \downarrow \binom{0}{\text{id}} & & \downarrow \binom{0}{\text{id}} & & \downarrow \phi \\
\dots & \longrightarrow & P^{-2} \oplus (M \otimes_B Q^{-2}) & \xrightarrow{\partial^{-2}} & P^{-1} \oplus (M \otimes_B Q^{-1}) & \longrightarrow & X \longrightarrow 0.
\end{array} \quad (3.5)$$

Putting (3.4) and (3.5) together we then get the following exact sequence of projective Λ -modules

$$L^\bullet = \dots \longrightarrow \left(\begin{smallmatrix} P^{-1} \oplus (M \otimes_B Q^{-1}) \\ Q^{-1} \end{smallmatrix} \right) \longrightarrow \left(\begin{smallmatrix} P^0 \oplus (M \otimes_B Q^0) \\ Q^0 \end{smallmatrix} \right) \xrightarrow{\binom{\partial^0}{\text{id}}} \left(\begin{smallmatrix} P^1 \oplus (M \otimes_B Q^1) \\ Q^1 \end{smallmatrix} \right) \longrightarrow \dots \quad (3.6)$$

with $\text{Ker} \left(\begin{smallmatrix} \partial^0 \\ d'^0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix} \right)_\phi$.

For each projective A -module P , $\text{Hom}_\Lambda(L^\bullet, \left(\begin{smallmatrix} P \\ 0 \end{smallmatrix} \right)) \cong \text{Hom}_A(P^\bullet, P)$ is exact, since P^\bullet is a complete projective resolution. For each projective B -module Q , since Q^\bullet is a complete projective resolution, $\text{Hom}_B(Q^\bullet, Q)$ is exact. Since $M \otimes_B Q$ is projective, $\text{Hom}_A(P^\bullet, M \otimes_B Q)$ is exact. Note that

$$\text{Hom}_\Lambda(L^\bullet, \left(\begin{smallmatrix} M \otimes_B Q \\ Q \end{smallmatrix} \right)) \cong \text{Hom}_A(P^\bullet, M \otimes_B Q) \oplus \text{Hom}_B(Q^\bullet, Q)$$

(here the direct sum only means that each term of the complex at the left hand side is a direct sum of terms of complexes at the right hand side, i.e., it does not mean a direct sum of complexes. In fact, the complex at the right hand side has differentials $\left(\begin{smallmatrix} \text{Hom}_A(d^i, M \otimes_B Q) & \text{Hom}_A(\sigma^i, M \otimes_B Q) \\ 0 & \text{Hom}_B(d'^i, Q) \end{smallmatrix} \right)$). By the canonical exact sequence of complexes

$$0 \longrightarrow \text{Hom}_A(P^\bullet, M \otimes_B Q) \xrightarrow{\binom{\text{id}}{0}} \text{Hom}_\Lambda(L^\bullet, \left(\begin{smallmatrix} M \otimes_B Q \\ Q \end{smallmatrix} \right)) \xrightarrow{\binom{0}{\text{id}}} \text{Hom}_B(Q^\bullet, Q) \longrightarrow 0$$

and the fundamental theorem of homological algebra we see that $\text{Hom}_\Lambda(L^\bullet, \left(\begin{smallmatrix} M \otimes_B Q \\ Q \end{smallmatrix} \right))$ is also exact. Therefore we conclude that L^\bullet is a complete Λ -projective resolution, and hence $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix} \right)_\phi$ is a Gorenstein-projective Λ -module.

Conversely, assume that $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix} \right)_\phi \in \mathcal{GP}(\Lambda)$. Then there is a complete Λ -projective resolution (3.6) with $\text{Ker} \left(\begin{smallmatrix} \partial^0 \\ d'^0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix} \right)_\phi$. Then we get an exact sequence (3.1) of projective B -modules with $\text{Ker } d'^0 = Y$, and the following exact sequence

$$V^\bullet = \dots \longrightarrow P^{-1} \oplus (M \otimes_B Q^{-1}) \longrightarrow P^0 \oplus (M \otimes_B Q^0) \xrightarrow{\partial^0} P^1 \oplus (M \otimes_B Q^1) \longrightarrow \dots \quad (3.7)$$

of projective A -modules with $\text{Ker } \partial^0 = X$. Since M_B is projective, it follows that $M \otimes_B Q^\bullet$ is exact. Since $\left(\begin{smallmatrix} \partial^i \\ d'^i \end{smallmatrix} \right)$ is a Λ -map, by (3.6) we know that ∂^i is of the form $\partial^i = \left(\begin{smallmatrix} d^i & 0 \\ \sigma^i & \text{id} \otimes_B d'^i \end{smallmatrix} \right)$, where

$\sigma^i : P^i \longrightarrow M \otimes_B Q^i$, $\forall i \in \mathbb{Z}$, and

$$P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

is a complex. By the canonical exact sequence of complexes

$$0 \longrightarrow M \otimes_B Q^\bullet \xrightarrow{\begin{pmatrix} 0 \\ \text{id} \end{pmatrix}} V^\bullet \xrightarrow{(\text{id}, 0)} P^\bullet \longrightarrow 0$$

and the fundamental theorem of homological algebra we see that P^\bullet is also exact.

From (3.6) we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_B Y & \longrightarrow & M \otimes_B Q^0 & \longrightarrow & M \otimes_B Q^1 \longrightarrow \cdots \\
& & \downarrow \phi & & \downarrow \begin{pmatrix} 0 \\ \text{id} \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ \text{id} \end{pmatrix} \\
0 & \longrightarrow & X & \longrightarrow & P^0 \oplus (M \otimes_B Q^0) & \longrightarrow & P^1 \oplus (M \otimes_B Q_1) \longrightarrow \cdots \\
& & \downarrow & & \downarrow (\text{id}, 0) & & \downarrow (\text{id}, 0) \\
0 & \longrightarrow & \text{Coker } \phi & \longrightarrow & P^0 & \xrightarrow{d^0} & P^1 \longrightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Thus $\phi : M \otimes_B Y \longrightarrow X$ is injective and $\text{Ker } d^0 \cong \text{Coker } \phi$. For each projective A -module P , since $\text{Hom}_\Lambda(L^\bullet, \begin{pmatrix} P \\ 0 \end{pmatrix}) \cong \text{Hom}_A(P^\bullet, P)$ and L^\bullet is a complete projective resolution, it follows that P^\bullet is a complete projective resolution, and hence $\text{Coker } \phi$ is a Gorenstein-projective A -module.

For each projective B -module Q , since P^\bullet is a complete projective resolution, it follows that $\text{Hom}_A(P^\bullet, M \otimes_B Q)$ is exact. Since L^\bullet is a complete projective resolution, it follows that

$$\text{Hom}_\Lambda(L^\bullet, \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}) \cong \text{Hom}_A(P^\bullet, M \otimes_B Q) \oplus \text{Hom}_B(Q^\bullet, Q)$$

is exact (again, here the direct sum does not mean a direct sum of complexes). By the same argument we know that $\text{Hom}_B(Q^\bullet, Q)$ is exact. It follows that Y is a Gorenstein-projective B -module. This completes the proof. \blacksquare

We remark that if Λ is Gorenstein, then in Theorem 3.1 $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \in \mathcal{GP}(\Lambda)$ implies $X \in \mathcal{GP}(A)$ (see [XZ], Corollary 3.3).

4. Main result

4.1. The aim of this section is to prove the following characterization of Gorenstein-projective Λ -modules, where Λ is the path algebra of a finite acyclic quiver over a finite-dimensional algebra. We emphasize that here Λ is not assumed to be Gorenstein.

Theorem 4.1. *Let Q be a finite acyclic quiver, and A a finite-dimensional algebra over a field k . Let $\Lambda = A \otimes_k kQ$, and $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ be a Λ -module. Then $X \in \mathcal{GP}(\Lambda)$ if and only if $X \in \text{Mon}(Q, A)$ and X satisfies the following condition (G), where*

(G) For each $i \in Q_0$, $X_i \in \mathcal{GP}(A)$, and the quotient $X_i / (\bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha) \in \mathcal{GP}(A)$.

Example 4.2. (i) Taking $Q = \bullet_n \longrightarrow \cdots \longrightarrow \bullet_1$ in Theorem 4.1 we get: a $T_n(A)$ -module $X = (X_i, \phi_i)$ is Gorenstein-projective if and only if each ϕ_i is injective, each X_i is a Gorenstein-projective A -module, and each $\text{Coker } \phi_i$ is a Gorenstein-projective A -module. Under the assumption that A is Gorenstein, this result was obtained in Corollary 4.1 of [Z].

(ii) Let Λ be the k -algebra given by quiver $\begin{array}{ccccc} & \lambda_3 & & \lambda_1 & \\ & \curvearrowright & & \curvearrowright & \\ \bullet_3 & \xrightarrow{\beta} & \bullet_1 & \xleftarrow{\alpha} & \bullet_2 \\ & & & & \lambda_2 \\ & & & & \curvearrowright \end{array}$ with relations $\lambda_1^2, \lambda_2^2, \lambda_3^2, \alpha\lambda_2 - \lambda_1\alpha, \beta\lambda_3 - \lambda_1\beta$. Then $\Lambda = A \otimes_k kQ = \begin{pmatrix} A & A & A \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, where Q is the quiver $\bullet_3 \longrightarrow \bullet_1 \longleftarrow \bullet_2$, and $A = k[x]/\langle x^2 \rangle$. Let k be the simple A -module, and $\sigma : k \hookrightarrow A$ the inclusion. Then by Theorem 4.1

$$X = (X_1 = A \oplus k, X_2 = k, X_3 = k, X_\alpha = \begin{pmatrix} 0 \\ \text{id} \end{pmatrix}, X_\beta = \begin{pmatrix} \sigma \\ \text{id} \end{pmatrix}) \in \mathcal{GP}(\Lambda);$$

while

$$Y = (Y_1 = A, Y_2 = k, Y_3 = k, Y_\alpha = \sigma = Y_\beta) \notin \mathcal{GP}(\Lambda).$$

4.2. Theorem 4.1 will be proved by using Theorem 3.1 and induction on $|Q_0|$, the number of vertices of Q .

We label Q_0 as $1, \dots, n$, such that if there is an arrow $\alpha : j \longrightarrow i$ in Q_1 , then $j > i$. Thus n is a source of Q . Denote by Q' the quiver obtained from Q by deleting vertex n , and by $\Lambda' = A \otimes_k kQ'$ the path algebra of Q' over A . Let $P(n)$ be the indecomposable projective (left) kQ -module at vertex n . Put $P = A \otimes_k \text{rad}P(n)$. Clearly P is a Λ' - A -bimodule and $\Lambda = \begin{pmatrix} \Lambda' & P \\ 0 & A \end{pmatrix}$. See (2.2).

Since kQ is hereditary, $\text{rad}P(n)$ is a projective kQ' -module, and hence $P = A \otimes_k \text{rad}P(n)$ is a (left) projective Λ' -module, and a (right) projective A -module (since as a right A -module, P is a direct sum of copies of A_A). This allows us to apply Theorem 3.1. For this, we write a Λ -module $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ as $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$, where $X' = (X_i, X_\alpha, i \in Q'_0, \alpha \in Q'_1)$ is a Λ' -module, and $\phi : P \otimes_A X_n \longrightarrow X'$ is a Λ' -map, whose explicit expression will be given in the proof of Lemma 4.4 below.

We will keep all these notations of $Q', \Lambda', P(n), P, X'$ and ϕ , throughout this section.

4.3. By a direct translation from Theorem 3.1 in this special case, we have

Lemma 4.3. Let $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$ be a Λ -module. Then $X \in \mathcal{GP}(\Lambda)$ if and only if X satisfies the following conditions:

- (i) $X_n \in \mathcal{GP}(A)$;
- (ii) $\phi : P \otimes_A X_n \longrightarrow X'$ is injective;
- (iii) $\text{Coker } \phi \in \mathcal{GP}(\Lambda')$.

For each $i \in Q'_0$, denote by $\mathcal{A}(n \rightarrow i)$ the set of the arrows from n to i ; and by $\mathcal{P}(n \rightarrow i)$ the set of paths from n to i . For an integer $m \geq 0$ and a module M , let M^m denote the direct sum of m copies of M .

Lemma 4.4. *Let $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ be a Λ -module. If X_β is injective for each $\beta \in Q'_1$, then $\phi : P \otimes_A X_n \rightarrow X'$ is injective if and only if X_α is injective, $\forall \alpha \in Q_1$, and $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p, \forall i \in Q'_0$.*

Proof. For $i \in Q'_0$, set $m_i = |\mathcal{P}(n \rightarrow i)|$. As a kQ' -module, $\text{rad}P(n)$ can be written as $\begin{pmatrix} k^{m_1} \\ \vdots \\ k^{m_{n-1}} \end{pmatrix}$ (please see (2.1) and 4.2), hence we have isomorphisms of Λ' -modules

$$P \otimes_A X_n \cong (\text{rad}P(n) \otimes_k A) \otimes_A X_n \cong \text{rad}P(n) \otimes_k X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix}.$$

Let $\mathcal{P}(n \rightarrow i) = \{p_1, \dots, p_{m_i}\}$. Then ϕ is of the form

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n-1} \end{pmatrix} : P \otimes_A X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix} \longrightarrow \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix},$$

where $\phi_i = (X_{p_1}, \dots, X_{p_{m_i}}) : X_n^{m_i} \rightarrow X_i$ (for the meaning of X_{p_i} please see 2.3). So ϕ is injective if and only if ϕ_i is injective for each $i \in Q'_0$; if and only if $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$, and X_p is injective, $\forall p \in \mathcal{P}(n \rightarrow i)$, from which and the assumption the assertion follows. ■

Lemma 4.5. *Let $X = \left(\frac{X'}{X_n}\right)_\phi$ be a monic Λ -module. Then*

- (1) *For each $i \in Q'_0$ there holds $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$;*
- (2) *$\phi : P \otimes_A X_n \rightarrow X'$ is injective;*
- (3) *$\text{Coker } \phi = (X_i / (\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p), \widetilde{X}_\alpha, i \in Q'_0, \alpha \in Q'_1)$, where for each $\alpha : j \rightarrow i$ in Q'_1 ,*

$$\widetilde{X}_\alpha : X_j / (\bigoplus_{q \in \mathcal{P}(n \rightarrow j)} \text{Im } X_q) \longrightarrow X_i / (\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p)$$

is the A -map induced by X_α .

Proof. By Lemma 4.4 and its proof it suffices to prove (1). For each $i \in Q'_0$, set $l_i = 0$ if $\mathcal{P}(n \rightarrow i)$ is empty, and $l_i = \max\{l(p) \mid p \in \mathcal{P}(n \rightarrow i)\}$ if otherwise, where $l(p)$ is the length of p . We prove (1) by using induction on l_i . If $l_i = 0$, then (1) trivially holds. Suppose $l_i \geq 1$. Let

$\sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}) = 0$ for $x_{n,p} \in X_n$. Since

$$\sum_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} \text{Im } X_p = \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha) = i}} X_\alpha \left(\sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q \right),$$

we have

$$\begin{aligned} 0 &= \sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}) = \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{n,\alpha}) + \sum_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} X_p(x_{n,p}) \\ &= \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{n,\alpha}) + \sum_{\substack{\beta \in Q'_1 \\ e(\beta)=i}} X_\beta \left(\sum_{q \in \mathcal{P}(n \rightarrow s(\beta))} X_q(x_{n,\beta q}) \right), \end{aligned}$$

by (m2) we know $X_\alpha(x_{n,\alpha}) = 0$ for $\alpha \in \mathcal{A}(n \rightarrow i)$, and $X_\beta \left(\sum_{q \in \mathcal{P}(n \rightarrow s(\beta))} X_q(x_{n,\beta q}) \right) = 0$ for $\beta \in Q'_1$ with $e(\beta) = i$. So $\sum_{q \in \mathcal{P}(n \rightarrow s(\beta))} X_q(x_{n,\beta q}) = 0$ by (m1). Since $l_{s(\beta)} < l_i$ for each $\beta \in Q'_1$ with $e(\beta) = i$, it follows from the inductive hypothesis that $X_q(x_{n,\beta q}) = 0$ for $\beta \in Q'_1$, $e(\beta) = i$ and $q \in \mathcal{P}(n \rightarrow s(\beta))$. This proves (1). \blacksquare

Lemma 4.6. *Let $X = \left(\frac{X'}{X_n} \right)_\phi$ be a monic Λ -module. Then $\text{Coker } \phi$ is a monic Λ' -module.*

Proof. We need to prove that for each $i \in Q'_0$, the Λ' -map

$$(\widetilde{X}_\alpha)_{\alpha \in Q'_1, e(\alpha)=i} : \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} (X_{s(\alpha)} / \left(\bigoplus_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q \right)) \longrightarrow X_i / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right)$$

is injective. For this, assume $\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \widetilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}}) = 0$, where $\overline{x_{s(\alpha),\alpha}}$ is the image of $x_{s(\alpha),\alpha} \in X_{s(\alpha)}$ in $X_{s(\alpha)} / \left(\bigoplus_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q \right)$. Then $\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) \in \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$. So there are $x_{n,p} \in X_n$ such that

$$\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = \sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}).$$

Thus

$$\begin{aligned} 0 &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}) \\ &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{\beta \in \mathcal{A}(n \rightarrow i)} X_\beta(x_{n,\beta}) - \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha \left(\sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,\alpha q}) \right) \\ &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,\alpha q}) - \sum_{\beta \in \mathcal{A}(n \rightarrow i)} X_\beta(x_{n,\beta}). \end{aligned}$$

Using the assumption on X we get $x_{s(\alpha),\alpha} = \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,\alpha q})$, i.e., $\overline{x_{s(\alpha),\alpha}} = 0$. \blacksquare

Lemma 4.7. *Let $X = \left(\frac{X'}{X_n} \right)_\phi$ be a monic Λ -module satisfying (G). Then*

$$(X_i / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right)) / \left(\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \widetilde{X}_\alpha \right)$$

is a Gorenstein-projective A -module, $\forall i \in Q'_0$.

Proof. Since $\bigoplus_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} \text{Im } X_p \subseteq \sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta$, it follows that

$$\begin{aligned}
\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \widetilde{X}_\alpha &= \left(\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } X_\alpha + \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\
&= \left(\sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta + \bigoplus_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} \text{Im } X_p \right) / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\
&= \left(\sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta \right) / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\
&\stackrel{(m2)}{=} \left(\bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta \right) / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right), \tag{4.1}
\end{aligned}$$

and hence the desired quotient is $X_i / \left(\bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta \right)$, which is Gorenstein-projective by (G). ■

Lemma 4.8. *Let $X = (X_n)_{\phi}$ be a monic Λ -module satisfying (G). Then for each $i \in Q'_0$, $X_i / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right)$ is a Gorenstein-projective A -module.*

Proof. We prove the assertion by using induction on l_i , which is defined in the proof of Lemma 4.5. If $i \in Q'_0$ with $l_i = 0$, then the assertion follows from (G).

Suppose $l_i \geq 1$. Since $\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \subseteq \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha$, we have the following exact sequence

$$0 \longrightarrow \left(\bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha \right) / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \longrightarrow X_i / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \longrightarrow X_i / \left(\bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha \right) \longrightarrow 0,$$

by (G) the term at the right hand side is Gorenstein-projective. It suffices to prove that the term at the left hand side is Gorenstein-projective. While by (4.1) it is $\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \widetilde{X}_\alpha$. By Lemma 4.6

each \widetilde{X}_α is injective, it follows that $\text{Im } \widetilde{X}_\alpha \cong X_j / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow j)} \text{Im } X_p \right)$, where $j = s(\alpha)$. Since $l_j < l_i$, it follows from the inductive hypothesis that $X_j / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow j)} \text{Im } X_p \right)$ is Gorenstein-projective. This completes the proof. ■

Lemma 4.9. *The sufficiency in Theorem 4.1 holds. That is, if $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ is a monic Λ -module satisfying (G), then X is Gorenstein-projective.*

Proof. Using induction on $n = |Q_0|$. The assertion clearly holds for $n = 1$. Suppose that the assertion holds for $n - 1$ with $n \geq 2$. It suffices to prove that X satisfies the conditions (i), (ii) and (iii) in Lemma 4.3.

The condition (i) is contained in (G); and the condition (ii) follows from Lemma 4.5(2). By Lemma 4.6 $\text{Coker } \phi$ is a monic Λ' -module; and by Lemmas 4.7 and 4.8 we know that $\text{Coker } \phi$ satisfies (G). It follows from the inductive hypothesis that the condition (iii) is satisfied. ■

Lemma 4.10. *Let $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ be a Λ -module with X_n a Gorenstein-projective A -module. Then $P \otimes_A X_n$ is a Gorenstein-projective Λ' -module, where P is defined in 4.2.*

Proof. Let $P(n)$ be the indecomposable projective kQ -module at vertex n . Writing $\text{rad}P(n)$ as a representation of Q' over k , we have $\text{rad}P(n) = (k^{m_i}, f_\alpha, i \in Q'_0, \alpha \in Q'_1)$, where $m_i = |\mathcal{P}(n \rightarrow i)|$ for each $i \in Q'_0$. By the construction of $P(n)$ we know that $\text{rad}P(n)$ has the following three properties:

- (1) each $f_\alpha : k^{m_{s(\alpha)}} \rightarrow k^{m_{e(\alpha)}}$ is injective;
- (2) for each $i \in Q'_0$ there holds $\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } f_\alpha = \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } f_\alpha$;
- (3) for each $i \in Q'_0$, $k^{m_i} / (\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } f_\alpha) \cong k^{|\mathcal{A}(n \rightarrow i)|}$ as k -spaces.

It follows that

$$P \otimes_A X_n \cong (\text{rad}P(n) \otimes_k A) \otimes_A X_n \cong \text{rad}P(n) \otimes_k X_n = (X_n^{m_i}, f_\alpha \otimes_k \text{id}_{X_n}, i \in Q'_0, \alpha \in Q'_1).$$

By (1), (2) and (3) we clearly see that $P \otimes_A X_n$ is a monic Λ' -module satisfying (G) (for example, by (3) we know that $X_n^{m_i} / (\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im}(f_\alpha \otimes_k \text{id}_{X_n})) \cong X_n^{|\mathcal{A}(n \rightarrow i)|}$ is a Gorenstein-projective A -module).

Now the assertion follows from Lemma 4.9. ■

4.4. Proof of Theorem 4.1 By Lemma 4.9 it remains to prove the necessity, i.e., if X is a Gorenstein-projective Λ -module, then X is a monic Λ -module satisfying (G). Using induction on $n = |Q_0|$. The assertion is clear for $n = 1$. Suppose that the assertion holds for $n - 1$ with $n \geq 2$. We write as $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$. Then X satisfies the conditions (i), (ii) and (iii) in Lemma 4.3.

By the condition (i) and Lemma 4.10 we know that $P \otimes_A X_n$ a Gorenstein-projective Λ' -module. Then by the conditions (ii) and (iii) we know that $X' \in \mathcal{GP}(\Lambda')$ since $\mathcal{GP}(\Lambda')$ is closed under extensions. By the inductive hypothesis X' is a monic Λ' -module satisfying (G), thus the following properties hold:

- (1) X_β is injective for each $\beta \in Q'_1$; and
- (2) X_i is Gorenstein-projective for each $i \in Q'_0$.

By (1), the condition (ii) and Lemma 4.4 we know that

- (3) X_α is injective for each $\alpha \in Q_1$; and
- (4) $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p, \forall i \in Q'_0$.

Since $\text{Coker } \phi = (X_i / (\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p), \widetilde{X}_\alpha, i \in Q'_0, \alpha \in Q'_1)$ is a Gorenstein-projective Λ' -module, it follows from the inductive hypothesis that the following properties hold:

- (5) for each $\alpha \in Q'_1$, \widetilde{X}_α is injective; and
- (6) $\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \widetilde{X}_\alpha = \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \widetilde{X}_\alpha, \forall i \in Q'_0$.

We first prove Claim 1: X satisfies (m2). In fact, suppose

$$\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = 0. \quad (*)$$

Since

$$\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{s(\alpha),\alpha}) + \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}),$$

it follows that

$$\begin{aligned} \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \widetilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}}) &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) + \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\ &\stackrel{\text{by} (*)}{=} - \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{s(\alpha),\alpha}) + \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) = 0. \end{aligned}$$

Then by (6) we have $\widetilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}}) = 0$; and by (5) we know $\overline{x_{s(\alpha),\alpha}} = 0$ for each $\alpha \in Q'_1$ with $e(\alpha) = i$. This means that there are $x_{n,q} \in X_n$ such that

$$x_{s(\alpha),\alpha} = \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,q}) \in \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q$$

for each $\alpha \in Q'_1$ with $e(\alpha) = i$. By (*) we have

$$0 = \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{n,\alpha}) + \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha \left(\sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,q}) \right).$$

By (4) we know that $X_\alpha(x_{n,\alpha}) = 0, \forall \alpha \in \mathcal{A}(n \rightarrow i)$, and $X_\alpha X_q(x_{n,q}) = 0, \forall \alpha \in Q'_1$ with $e(\alpha) = i$ and $q \in \mathcal{P}(n \rightarrow s(\alpha))$. Thus $X_\alpha(x_{s(\alpha),\alpha}) = 0, \forall \alpha \in Q_1$ with $e(\alpha) = i$. This proves Claim 1.

We now prove Claim 2: $X_i / \left(\bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta \right)$ is a Gorenstein-projective A -module for each $i \in Q_0$.

In fact, since $\text{Coker } \phi$ is a Gorenstein-projective Λ' -module, by the inductive hypothesis we know that

$$(X_i / \left(\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right)) / \left(\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \widetilde{X}_\alpha \right)$$

is a Gorenstein-projective A -module: it is exactly the desired module by (4.1).

Now, (3) and Claim 1 mean that X is a monic Λ -module; and (2), the condition (i), together with Claim 2 mean that X satisfies (G). This completes the proof. \blacksquare

5. Applications

We include some applications of Theorem 4.1. Let Λ be the path algebra of finite acyclic quiver Q over finite-dimensional algebra A . Recall that $\text{Mon}(Q, A)$ denotes the full subcategory of $\text{Rep}(Q, A)$ consisting of the monic representations of Q over A .

5.1. As a consequence of Theorem 4.1, we get the following characterization of self-injectivity.

Theorem 5.1. *Let A be a finite-dimensional algebra. Then the following are equivalent:*

- (i) A is self-injective;
- (ii) For any finite acyclic quiver Q , there holds $\mathcal{GP}(A \otimes_k kQ) = \text{Mon}(Q, A)$;
- (iii) There is a finite acyclic quiver Q , such that $\mathcal{GP}(A \otimes_k kQ) = \text{Mon}(Q, A)$.

Proof. (i) \implies (ii): If A is self-injective, then every A -module is Gorenstein-projective, and hence (ii) follows from Theorem 4.1. The implication (ii) \implies (iii) is clear.

(iii) \implies (i): Take a sink of Q , say vertex 1, and consider the representation X of Q over A , where $X_1 = \text{Hom}_A(A, k)$ and $X_i = 0$ if $i \neq 1$. Then X is a monic Λ -module, and hence by assumption it is Gorenstein-projective. So we have a complete Λ -projective resolution

$$\cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

with $X \cong \text{Ker } d^0$. By taking the 1-st branch we get an exact sequence

$$\cdots \longrightarrow P_1^{-1} \longrightarrow P_1^0 \xrightarrow{d_1^0} P_1^1 \longrightarrow \cdots$$

with $\text{Ker } d_1^0 \cong \text{Hom}_A(A, k)$. Note that each P_1^i is a projective A -module. Thus injective A -module $\text{Hom}_A(A, k)$ is projective, i.e., A is self-injective. \blacksquare

Example 5.2. Taking $Q = \bullet_n \longrightarrow \cdots \longrightarrow \bullet_1$ in Theorem 5.1 we get: A is a self-injective algebra if and only if the Gorenstein-projective $T_n(A)$ -modules are exactly the monic $T_n(A)$ -modules. Under the assumption that A is Gorenstein, this result was obtained in Theorem 4.4 of [Z].

Let $D^b(\Lambda)$ be the bounded derived category of Λ , and $K^b(\mathcal{P}(\Lambda))$ the bounded homotopy category of $\mathcal{P}(\Lambda)$. By definition the singularity category $D_{sg}^b(\Lambda)$ of Λ is the Verdier quotient $D^b(\Lambda)/K^b(\mathcal{P}(\Lambda))$. If Λ is Gorenstein, then there is a triangle-equivalence $D_{sg}^b(\Lambda) \cong \underline{\mathcal{GP}}(\Lambda)$, where $\underline{\mathcal{GP}}(\Lambda)$ is the stable category of $\mathcal{GP}(\Lambda)$ modulo $\mathcal{P}(\Lambda)$ (see [Hap], Theorem 4.6; also [Buch], Theorem 4.4.1). Note that if A is Gorenstein, then $\Lambda = A \otimes_k kQ$ is Gorenstein, by Proposition 2.2 in [AR], which claims that $A \otimes_k B$ is Gorenstein if and only if A and B are Gorenstein. So we have

Corollary 5.3. *Let A be a finite-dimensional Gorenstein algebra, and Q a finite acyclic quiver. Let $\Lambda = A \otimes_k kQ$. Then there is a triangle-equivalence $D_{sg}^b(\Lambda) \cong \underline{\mathcal{GP}}(\Lambda)$.*

5.2. Before giving the next application we recall the tensor product of two finite quivers. Let Q and Q' be finite quivers (not necessarily acyclic). By definition the tensor product $Q \otimes Q'$ is the quiver with

$$(Q \otimes Q')_0 = Q_0 \times Q'_0, \quad \text{and} \quad (Q \otimes Q')_1 = (Q_1 \times Q'_0) \bigcup (Q_0 \times Q'_1).$$

More explicitly, if $\alpha : i \longrightarrow j$ is an arrow of Q , then for each vertex $t' \in Q'_0$ there is an arrow $(\alpha, t') : (i, t') \longrightarrow (j, t')$ of $Q \otimes Q'$; and if $\beta' : s' \longrightarrow t'$ is an arrow of Q' , then for each vertex $i \in Q_0$ there is an arrow $(i, \beta') : (i, s') \longrightarrow (i, t')$ of $Q \otimes Q'$.

Let $A = kQ/I$ and $B = kQ'/I'$ be two finite-dimensional k -algebra, where Q and Q' are finite quivers (not necessarily acyclic), I and I' are admissible ideals of kQ and kQ' , respectively. Then

$$A \otimes_k B \cong k(Q \otimes Q')/I \square I',$$

where $I \square I'$ is the ideal of $k(Q \otimes Q')$ generated by $(I \times Q'_0) \cup (Q_0 \times I')$ and the following elements

$$(\alpha, t')(i, \beta') - (j, \beta')(\alpha, s'),$$

where $\alpha : i \rightarrow j$ is an arrow of Q , and $\beta' : s' \rightarrow t'$ is an arrow of Q' . See for example [L]. Note that $I \square I'$ may not be zero even if $I = 0 = I'$. Thus we have the following

Fact: $A \otimes_k B$ is hereditary (i.e., $I \square I' = 0$) if and only if either

- (i) $A \cong k^{|Q_0|}$ as algebras, and $I' = 0$; or
- (ii) $B \cong k^{|Q'_0|}$ as algebras, and $I = 0$.

5.3. We describe when Λ is hereditary via monic Λ -modules.

Theorem 5.4. *Let Λ be the path algebra of finite quiver Q over A , where Q is acyclic with $|Q_1| \neq 0$, and A is a finite-dimensional basic algebra over an algebraically closed field k . Then $\mathcal{P}(\Lambda) = \text{Mon}(Q, A)$ if and only if Λ is hereditary.*

Proof. Without loss of generality we may assume that A is connected (an algebra is connected if it can not be a product of two non-zero algebras).

If $\Lambda = A \otimes_k kQ$ is hereditary, then by the fact above and the assumption of Q we have $A = k$, and hence $\text{Mon}(Q, k) = \mathcal{GP}(kQ)$ by Theorem 4.1. It follows that

$$\text{Mon}(Q, A) = \text{Mon}(Q, k) = \mathcal{GP}(kQ) = \mathcal{P}(kQ) = \mathcal{P}(\Lambda).$$

Conversely, if $A \neq k$, then A is not semi-simple since A is assumed to be connected and basic and k is assumed to be algebraically closed. It follows that there is a non-projective A -module M . Take a sink of Q , say vertex 1, and consider Λ -module $X = M \otimes_k P(1)$, where $P(1)$ is the simple projective kQ -module at vertex 1. Then as a representation of Q over A we have $X = (X_i, i \in Q_0)$ with $X_1 = M$ and $X_i = 0$ for $i \neq 1$. It is clear that $X \in \text{Mon}(Q, A)$, but $X \notin \mathcal{P}(\Lambda)$. ■

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